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Radiation patterns of linear arrays

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OF

LINEAR ARRAYS

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by

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PREFACE

The purpose of this paper is to consider in general the sum and difference patterns of symmetric linear arrays of 2n elements, showing, where mathematically feasible, the relationship existing between them, and to determine the behavior of the phase and amplitude characteristics resulting from their superposition.

The sum pattern is defined in the same sense as the broadside pattern with uniform phase distribution; the difference pattern differs from the broadside pattern in that the excitation of all radiators on one side of the array is reversed in phase.

Three types of array as regards amplitude distribution are exemplified. The array in which the amplitude of the exciting currents in its radiating elements are proportional to the binomial coefficients and the uniform array are chosen as examples of the extremes, while the array with an amplitude distribution which produces equal side lobes with optimum beam width serves as the intermediate array.

In particular the patterns are calculated and plotted for an eight-element array of unspecified spacing, and a specific example is given for an electrical spacing of 252 degrees.

It is shown that in general the characteristics of the difference pattern are predictable qualitatively from a knowledge of the sum characteristics.

By considering the two symmetric halves of the array as separate receiving antennas, and by suitably combining their received signals to form sum and difference voltages analagous to the sum and difference patterns, it is shown that the resultant phase characteristic indicates the direction of approach of the exciting wave; and further, that no directional ambiguity is possible for spacing of the elements less than a full wave length.

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LIST OF ILLUSTRATIONS

TABLE OF SYMBOLS

A,a,B	Amplitude coefficients
d	linear spacing of adjacent radiators
E	Radiation pattern as a function of θ
F	Radiation pattern as a function of u
G	Radiation pattern as a polyinomial in x or y
I	coefficient proportional to exciting current
i	$\sqrt{-1}$
k	(subscript) designates general term
m,n	integers
n	usually, the number of pairs of radiators
r	a numerical ratio
S	sum of series; space factor
s	an integer
T	Tchebyscheff polynomial
u	universal angle for radiation patterns $= \frac{\pi d}{\lambda} \sin\theta$
v,w	substitute variables in reduced equations
x	cos u
y	sin u
z	Independent variable = cos φ when $\lvert z\rvert \leq 1$
z_o	a parameter, or scale factor
β	phase constant $= \frac{2\pi}{\lambda}$
γ	2 cos u
Δ	(subscript), denotes difference pattern
θ	Angle to array normal in plane containing array
ϑ	Phase angle
φ	general angle - arc cos z

σ (subscript), denotes sum pattern

ψ a phase angle

Subscripts N, 2n-1 etc. denote degree of polynomial or degree of term to which a coefficient belongs.

CHAPTER I

THE GENERAL ARRAY

Since the radiation pattern of several identical and similarly oriented radiators is expressible as the product of the radiation pattern of a typical element and the space factor of the array, it will suit our present purpose to assume a similar array of non-directive sources, whence the terms radiation pattern and space factor become synonomous. Further for simplicity we use the electric field intensity rather than radiation intensity at remote points equidistant from the array center to represent the radiation pattern. The contribution to the electric field strength at a distant point by a single radiating element can be expressed as the product of a complex amplitude coefficient, a time varying term, and a translation factor, by means of which the given source is reduced to an equivalent virtual source at the array center. Suppressing the time variation, this expression takes the form

$$E_k = A_k \epsilon^{i\beta r_k \sin\theta} \tag{1}$$

in which $A_k = a_k \epsilon^{i\vartheta_k}$ contains the amplitude and phase of excitation,

β = phase constant = $\frac{2\pi}{\lambda}$, and $r_k \sin\theta$

is the projection of the line from the array center to point P upon the line from the source position to point P.

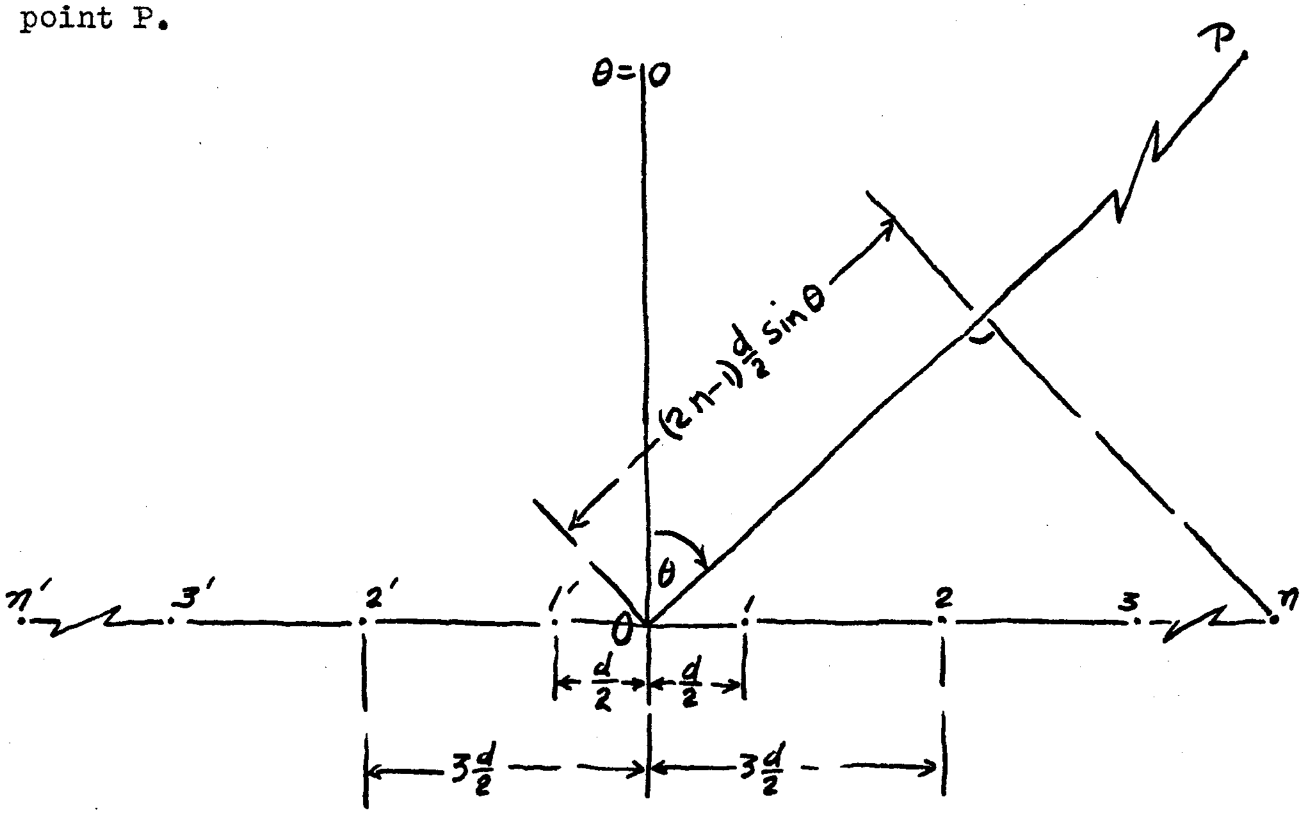

Figure 1
The symmetric linear array of 2n elements.

Consider the symmetric linear array of 2n equi-spaced non-directive radiators symmetrically disposed about the array center as shown in fig.1. In the general case the strength of opposite sources will be taken to be equal. The sources 1, 2, 3,-----n will be excited in reference phase; the elements 1^1, 2^1, 3^1,----n^1 will either be excited in reference phase or in anti-phase.

Then by analogy to (1) we write for the contribution to the electric field intensity by the pair kk^1

$$E_{KK'}(\theta) = a_K\left[\epsilon^{i\beta(2K-1)\frac{d}{2}\sin\theta} \pm \epsilon^{-i\beta(2K-1)\frac{d}{2}\sin\theta}\right] \tag{2}$$

or, introducing a new variable, $u = \pi\frac{d}{\lambda}\sin\theta$,

$$F_{KK'}(u) = a_K\left[\epsilon^{i(2K-1)u} \pm \epsilon^{-i(2K-1)u}\right]. \tag{3}$$

Taking first the uniform distribution of phase and summing (3) over k we obtain as the expression for the <u>sum pattern</u>

$$F_{\sigma}(u) = I_1\cos u + I_2\cos 3u + I_3\cos 5u + \cdots I_n\cos(2n-1)u \tag{4}$$
$$= \sum_{K=1}^{n} I_K\cos(2K-1)u.$$

The <u>difference pattern</u> resulting from the anti-phase distribution becomes

$$F_{\Delta}(u) = i\left[I_1\sin u + I_2\sin 3u + I_3\sin 5u + \cdots I_n\sin(2n-1)u\right]$$
$$= i\sum_{K=1}^{n} I_K\sin(2K-1)u \tag{5}$$

where, as in (4), the substitution I_k has been made for $2\,a_k$.

For most applications, neither the exponential (3) nor Fourier series (4)(5) representation affords a sufficient amount of flexibility, hence, we set up a third method of representing the directivity functions, viz. their expansion as polynomials in Cos u and Sin u.

A useful expansion for Cos n u arising from the expansion of log $(1-2r \text{ Cos } u+r^2)$ which contains no term in Sin u is[1]

$$2 \operatorname{Cos} nu = \gamma^n - n\gamma^{n-2} + \frac{n(n-3)}{2!}\gamma^{n-4} - \cdots \\ \cdots + (-1)^s \frac{n(n-s-1)(n-s-2)\cdots(n-2s+1)}{s!}\gamma^{n-2s} + \cdots \tag{6}$$

where $\gamma = 2 \cos u$.

Since we are restricted by symmetry of the array to an even number of radiators we will need only odd values of n in the expansion of sin nu, hence we can use the symmetric form

$$\operatorname{Sin} nu = ny - \frac{n(n^2-1^2)}{3!}y^3 + \frac{n(n^2-1^2)(n^2-3^2)}{5!}y^5 - \cdots \tag{7}$$

where y = sin u; n is an odd integer.

Using (6) we have for the sum pattern

$$G_\sigma(x) = A_{2n-1}x^{2n-1} + A_{2n-3}x^{2n-3} + \cdots \tag{8}$$

where X = Cos u,

and the A's contain the I's, and the numerical coefficients arising from the expansion.

In similar manner we represent the difference pattern by

$$G_{\Delta}(y) = B_{2n-1}\, y^{2n-1} + B_{2n-3}\, y^{2n-3} + \cdots \qquad (9)$$

where $y = \sin u$.

Now that we have general expressions for the radiation pattern for the symmetric 2n array described, we will attempt a limited analysis of the types of patterns obtainable as a function of the variables which are not yet specified. In this connection it is emphasized that the use of u as the primary variable instead of θ the true variable, in no way implies that d or λ is specified. On the contrary, u may be considered to be the primary variable and $\frac{\pi d}{\lambda}$ as the scale factor which determines the range of u for a specified range of θ and specified number of elements in the array. In the following analysis we assume that:

a) $-\pi \leq \theta \leq \pi$

b) The aperture is finite, hence $(2n-1)d \leq$ a specified number,

c) λ is specified (or alternately, changes in λ can be interpreted as variations in u),

d) No phase changes in the exciting currents other than the 180 degree shift in the case of the difference pattern are allowable.

It will be appreciated that, although the above assumptions reduce the problem to a consideration of the amplitude distribution among the n pairs of elements, the number of patterns which can be obtained is still infinite, and many methods have been devised for shaping the radiation pattern to a specified pattern by manipulation of both phase and amplitude distributions in the aperture (2,3,4,6).

We will consider three general types of amplitude distribution whose sum characteristics are well known, and examine the type of difference pattern which accompanys each. First we consider the uniform distribution as one of the extremes. This pattern is well known for its narrow primary lobe, is notorious in application for its high side lobe level, yet it represents maximum power gain obtainable with a given aperture. The binomial array serves as the other extreme. It has no side lobes on the range $|u| < \frac{\pi}{2}$, has a very broad primary lobe, and has the practical disadvantages of requiring relatively high currents in the center elements and of having low power gain. As an intermediate type of pattern we consider a compromise pattern based upon the suitably restricted Tchebyscheff polynomial (3) which has side lobes of equal maximum values at a specified level below that of the principal lobe.

In general, for simplicity, we shall normalize the currents so that

$$\sum I_k = 1 ,$$

CHAPTER II

THE LINEAR ARRAY WITH UNIFORM DISTRIBUTION OF AMPLITUDE

For the sum pattern we take $a_k = a_i = a$ and write (3) as the geometric progression

$$F_\sigma(u) = a\Big[\epsilon^{-i(2n-1)u} + \epsilon^{-i(2n-3)u} + \cdots\cdots \epsilon^{-iu} \cdots + \epsilon^{iu} + \epsilon^{i3u} + \cdots\cdots + \epsilon^{i(2n-1)u}\Big] \qquad (10)$$

whose sum reduces to

$$F_\sigma(u) = \frac{I}{2}\,\frac{\sin 2nu}{\sin u}, \qquad (11)$$

upon substituting I/2 for a, which is consistent with (4) and (5), The roots are readily found to be at $\pm\frac{N\pi}{2n}$, with the value of the indeterminate,

$$F_\sigma(0) = nI, \text{ and } F_\sigma(\pi) = -nI.$$

To determine the positions of the subsidiary lobes we have, by differentation

$$F_\sigma'(u) = \frac{I}{2}\left[\frac{2n \sin u \cos 2nu - \cos u \sin 2nu}{\sin^2 u}\right]$$

$$= 0 \text{ for } 2n\tan u = \tan 2nu \qquad (12)$$

which may be solved graphically, or alternately, if the degree is not too high we can expand (4) as a polynomial in $X = \cos u$ and solve by any of the several methods suitable to the degree of the polynomial.

For the difference pattern we again use (3) and write the exponential series

$$F_\Delta(u) = \frac{I}{2}\left\{-\left[\epsilon^{-i(2n-1)u} + \epsilon^{-i(2n-3)u} + \cdots + \epsilon^{-iu}\right] + \cdots + \epsilon^{iu} + \epsilon^{i3u} + \cdots + \epsilon^{i(2n-1)u}\right\}. \tag{13}$$

Summing separately the terms with positive and negative exponents we have for the partial sums

$$S^{+} = \frac{I}{2}\left[\epsilon^{inu}\frac{\sin nu}{\sin u}\right] \text{ and } S^{-} = \frac{I}{2}\left[-\epsilon^{-inu}\frac{\sin nu}{\sin u}\right]$$

which in combined form becomes

$$F_\Delta(u) = iI\,\frac{\sin^2 nu}{\sin u}. \tag{14}$$

To find the position of the nulls reqrite (14) as

$$|F_\Delta(u)| = I \sin nu\left(\frac{\sin nu}{\sin u}\right).$$

The value of the indeterminate is nullified by the zero factor, hence the nulls occur at $u = 0,\ \pm\frac{N\pi}{n}$.

$$|F_\Delta'(u)| = I\left[n\cos nu\left(\frac{\sin 2nu}{\sin u}\right) - \cos u\left(\frac{\sin nu}{\sin u}\right)^2\right]. \tag{15}$$

The value of the derivative at u = 0 is of particular interest since the slope of the function at zero is a measure of the"sharpness" of the null and affords a basis for comparing several arrays as to error sensitivity. From (15) we have

$$|F_\Delta'(0)| = I(2n^2-n^2) = In^2$$

$$= n \qquad \text{(for normalized currents)}$$

again, the positions of the minor lobes may be found either from the solution of the corresponding polynomial expression or from the graphic solution of

$$2n \tan u = \tan nu. \tag{16}$$

It is interesting (and profitable if plotting is contemplated) to note the relationship of the ratio of the difference to sum pattern. Re-writing (11) as

$\frac{I}{2} \frac{2 \sin nu \cos nu}{\sin u}$, we obtain

$$\left|\frac{F_\Delta(u)}{F_\sigma(u)}\right| = \tan nu. \tag{17}$$

Again noting the form of (14), we see that the null at zero and $\pm\frac{N\pi}{n}$ are single valued (since they arise from the product of a non-zero indeterminate by zero) whereas the remaining nulls arising from the double roots will be broad. The phase of the radiation will lag reference phase by $\pi/2$ for $-\pi \leq u$, and will lead reference phase by $\pi/2$ for $u \leq \pi$. Since this range of u corresponds to full wave spacing we would not expect, in the practical case, any phase change on the range of θ except at $\theta = u = 0$. This would be true in particular if the array is backed by a ground plane, which restricts the range of θ .

CHAPTER III

THE BINOMIAL ARRAY

To determine the nature of the pattern which results from a current distribution in which the amplitudes are proportional to the binomial coefficients consider (4) in the form

$$F_\sigma(u) = \cos mu + I_{n-1}\cos(m-2)u + I_{n-2}\cos(m-4)u + \cdots$$

where $m = 2n-1$ and $I_n = I_1 = 1$

Expanding the first few terms we obtain

$$\begin{aligned} F_\sigma(u) = \; & 2^{m-1}\cos^m u - \frac{2}{1!}^{m-3} m \cos^{m-2} u + \frac{2}{2!}^{m-5} m(m-3)\cos^{m-4} u - \cdots \\ & + 2^{m-3} I_{n-1}\cos^{m-2} u - I_{n-1}\frac{2}{1!}^{m-5}(m-2)\cos^{m-4} u + \cdots \\ & + I_{n-2} \cos^{m-4} u - \cdots \\ \hline = \; & 2^{m-1}\cos^m u - 2^{m-3}(m - I_{n-1})\cos^{m-2} u \cdots \\ & + 2^{m-5}\left[\frac{m(m-3)}{2} - (m-2)I_{n-1} + I_{n-2}\right]\cos^{m-4} u + \cdots \end{aligned}$$

The condition that the second coefficient vanish is $I_{n-1} = m$

The additional condition that the third coefficient vanish is $I_{n-2} = \frac{m(m-1)}{2}$.

By this stepwise process it is shown by induction that if the I_s are the binomial coefficients then all coefficients vanish in the summation except the coefficient of the term of highest degree.

Thus for the binomial distribution where the amplitudes in the outermost radiators are unity the radiation pattern is represented by

$$F_\sigma(u) = 2^{2n-2} \cos^{2n-1} u, \tag{18}$$

the first zero of which occurs at $u = \pm \pi/2$.

$$F_\sigma'(u) = 0 \qquad \text{for} \quad \sin u \cos^{2n-2} u = 0, \ i.e.$$
$$\text{for} \quad u = 0, \pm N\pi$$

When the binomial coefficients are substituted in the expression for the difference pattern no such simple result is obtained. The coefficients may be represented in the general case by cumbersome summations which are of little value in analysis. The analysis in this case will be made for a specified value of n.

CHAPTER IV

AN INTERMEDIATE ARRAY

We take as a pattern intermediate to the uniform and binomial patterns a pattern calculated by the method described by Dolph [3]. A complete discussion of the general case is presented in the reference and will not be duplicated here.

Since the general expression for the sum pattern involve triple summations an analysis of the difference pattern employing the primary distribution would be extremely tedious if not impossible. As in the case of the binomial array, the discussion of the difference pattern will be limited to the results of a specific example.

CHAPTER V

TYPICAL CALCULATIONS

The eight element array with spacing and frequency unspecified will be used for representative calculations.

1. Uniform distribution of amplitude

The nulls are found from (11) to be at $u_N^{\circ} = \frac{N\pi}{8}$

i.e. $u^{o} = 22.5, 45, 67.5, 90$ --- degrees.

To find the points of maximum radiation we take the derivative of (4), viz.

$$F_{\sigma}'(u) = -(\sin u + 3\sin 3u + 5\sin 5u + 7\sin 7u) \qquad \text{where } I_k = 1$$

and expand in the form

$$G_{\sigma}'(y) = 448\,y^7 - 864\,y^5 + 504\,y^3 - 84\,y \quad \text{where } y = \sin u.$$

By equating to zero and removing the common factor y, we solve as a reduced cubic in $V = y^2$. This gives, for the maxima

$$\hat{u} = 0,\ 32^{o}\text{-}21.4,\ 55^{o}\text{-}38.4,\ 78^{o}\text{-}34,\ 101^{o}\text{-}26, 124^{o}\text{-}21.6, \text{ etc.}$$

In this case the values of the function at these and at intermediate points were calculated in conjunction with the difference pattern using the tangent relationship, (17).

For the difference pattern the nulls are found at

$u_n^o = 0, \pm \frac{N\pi}{4}$

$u^o = 0, 45, 90, 135, 180$ - - degrees

To locate the maxima we again expand in a power series and obtain the derivative equation

$$|F_\Delta'(u)| = 16 \cos u(1 - 15y^2 + 40y^4 - 28y^6) = 0.$$

from $\cos u = 0$ we find $\hat{u}_1 = \pm \pi/2$.

By removing the root at $y^2 = 1/2$ we solve the remaining quadratic in y^2 and locate all the maxima at

$$\pm\hat{u} = 16^o - 54.8,\ 45^o,\ 66^o - 43.4,\ 90^o \text{ and at } (180^o - \hat{u})$$

Values of the sum and difference pattern calculated by five-place logarithms for every five degrees of u and at all critical points are shown on curves Fig. (3) and Fig.(4), respectively for normalized amplitude values.

2. The eight element binomial array.

For the binomial array the sum pattern has been shown to be

$$F_\sigma(u) = 2^6 \cos^7 u$$

with a principal in phase lobe at zero and an identical lobe of opposite phase at $u = \pi$.

1338 The nulls at $\pm \pi/2$ are accompanied by a 180 degree phase change.

For the difference pattern (5) becomes

$$|F_\Delta(u)| = 35 \sin u + 21 \sin 3u + 7 \sin 5u + \sin 7u$$

which expands into

$$|G_\Delta(y)| = 4(35y - 70y^3 + 56y^5 - 16y^7),$$

an equation whose only real root less than unity is $y \equiv \sin u = 0$. Hence, the only nulls occur at $u^0 = 0, \pi$.

To obtain the maxima of the pattern we again have a general cubic in y^2 to solve.

$$|F_\Delta'(u)| = 28 \cos u \,(5 - 30y^2 + 40y^4 - 16y^6) = 0$$

By synthetic division we find $y = .48116$, hence the position of the maxima are

$$\pm \hat{u} = \frac{1}{2},\ 28^\circ - 45.7,\ 131^\circ - 14.3.$$

The values of the difference pattern shown on Fig. (3) were obtained by straightforward calculation by five-place logarithms using its Fourier series expression.

3. The intermediate array.

For the intermediate array we expand (3) in the form

$$G_\sigma(x) = x\left\{64x^6 I_4 - (112 I_4 - 16 I_3)x^4 + (56 I_4 - 20 I_3 + 4 I_2)x^2 \cdots \right.$$
$$\left. -(7 I_4 - 5 I_3 + 3 I_2 - I_1)\right\}$$

where $X = \operatorname{Cos} u$ and hence $-1 \leq X \leq 1$

If we now take as the definition of the Tchebyscheff polynomial

$T_N(z) = \operatorname{Cos}(N \operatorname{arc} \operatorname{Cos} z)$ or specifically,

$T_7(z) = \operatorname{Cos}(7 \operatorname{arc} \operatorname{Cos} z)$, anticipating the need for a polynomial of seventh degree, and let $z = \operatorname{Cos} \varphi$ we have

$T_7(z) = \operatorname{Cos} 7\varphi$ which, upon expansion becomes

$$T_7(z) = z(64z^6 - 112z^4 + 56z^2 - 7).$$

Now for $-1 \leq z \leq 1$ this expression is simply $\operatorname{Cos} 7\varphi$, however, if we remove the restriction on z and let $-\infty \leq z \leq \infty$ it goes off toward $+\infty$ as z increases beyond $z = +1$ and, being of odd degree goes toward $-\infty$ as z is decreased below $z = -1$. Hence, we deduce, that for $z > 1$ we can find a value $z = z_0$ such that

$T_7(z_0) = R =$ any desired number > 1 .

If we then relate the value of the function at z_0 to our desired primary lobe at $u=0$ $[x=+1]$, the minor lobes will be represented by the value of the polynomial between $z=-1$ and $z=+1$. Hence, the principal lobe will be R times as large as the equal valued minor lobes.

We will now assume that the minor lobes should be not less than 20 db. below the principal lobe, i.e.

$$T_7(z_0) = z_0(64z_0^6 - 112z_0^4 + 56z_0^2 - 7) \geq 10$$

We try, by synthetic division, a value $z = z_0 = 1.1$ and find the remainder to be essentially 11.177, which is equivalent to 20.98 db. and will be taken to be satisfactory. Thus we decide to restrict z to the range

$-z_0 \leq z \leq z_0$

We desire to substitute for z a variable which will range between $-z_0$ and $+z_0$ while X ranges from -1 to 1.

By linear transformation we have

$$\frac{X-1}{2} = \frac{z-z_0}{2z_0} \qquad \text{whence}$$

$z = z_0X$ is the desired substitution.

The behavior of the polynomial is then

$$T_7(z_0) = R$$

$$T_7(z_0X) = \begin{cases} (z_0X)\{64(z_0X)^6 - 112(z_0X)^4 + 56(z_0X)^2 - 7\} & -1 \leq X \leq 1 \\ \cos 7\varphi & |X| \leq \frac{1}{z_0} \end{cases}$$

By equating termwise $T_7(z_0 x)$ and $G_\sigma(x)$ we obtain the following relations:

$I_4 = z_0^7$	$= 1.9487$	or	$.17435$
$I_3 = 7(I_4 - z_0^5)$	$= 2.3674$		$.21175$
$I_2 = 5I_3 - 14I_4 + 14z_0^3$	$= 3.1892$		$.28533$
$I_1 = 3I_2 - 5I_3 + 7I_4 - 7z_0$	$= 3.6714$		$.32851$
	11.1768		$.99994 \sim 1.000$

From the relations ,

$$z_0 x = z = \cos\varphi$$

we determine the position of the nulls and lobes from

$$u = \cos^{-1}\left(\frac{\cos\varphi}{z_0}\right).$$

$$\pm u^\circ = 27.^\circ6\ ;\ 44.^\circ7\ ;\ 66.^\circ8\ ;\ 90^\circ \text{ and } (180 - u^\circ)$$

$$\pm \hat{u} = 0;\ 35.^\circ0\ ;\ 54.^\circ6\ ;\ 78.^\circ3 \text{ and } (180 - \hat{u}).$$

Because of the complexity of the coefficients, no attempt was made to locate the nulls and lobes of the difference pattern having the above current distribution other than by plotting. The plot shows maxima at, 28.5 deg. and 66.5 deg., a null at zero, and a minimum at 45 deg.

CHAPTER VI

DISCUSSION AND CONCLUSION

The plot of the sum patterns in Fig.(3) gives no unexpected information. The curves demonstrate graphically the well known principle that any distribution which tends to suppress the side lobe level results in the broadening of the main forward lobe. Recalling that the gain of the array is roughly inversely proportional to the solid angle of the main lobe, we also note that side lobe suppression is obtainable only at a sacrifice in gain. We observe that the pattern with amplitude distribution based upon the Tchebyscheff polynomial is intermediate to the patterns with uniform and binomial distributions. That this is generally true is demonstrated by Dolph [3] who shows that the binomial distribution results from the choice of $z_0 = \infty$, and the uniform distribution results when $z_0 = 1$.

A study of the curves in Fig.4 reveals a similar behavior for the difference patterns, the intermediate pattern being intermediate as regards slope at the origin, and the magnitude and position of the first lobe. Nulls, other than at 0 and π , occur only in the case of uniform distribution. In general it can be shown that for any progressive decreasing taper in amplitude there can be no null on $0 < |u| < \pi$.

Tabulations of comparative data for the arrays considered in the foregoing examples are given in tables

I and II. All magnitudes are normalized in that the sum of all currents in the array is equal to unity. The angles indicated are values of u.

Table I - Sum patterns for 8-element array

Distribution	Half Power Beam Width	Beam width between nulls	First minor lobe position	magnitude
Uniform	19.8 deg	45 deg.	32.3deg	0.229
Optimum	22.4 "	55.2 "	35 "	0.090
Binomial	35.8 "	180 "	---	---

Table II - Difference patterns for 8-element array

Distribution	Principal lobe position	magnitude	First minor lobe position	magnitude	slope at $u=0$
Uniform	16.9 deg	0.737	66.7 deg	0.271	4.00
Optimum	19.5 "	0.680	66.8 "	0.283	3.46
Binomial	28.8 "	0.647	---	---	2.19

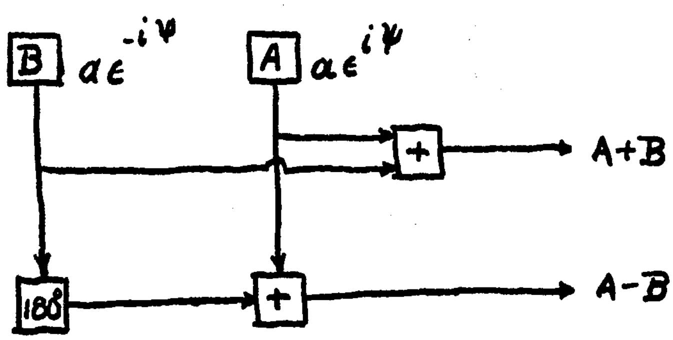

Figure 2
Block diagram of receiving array

Let us now consider the symmetric halves of our array as the two sources A and B as shown in fig. (2) with external excitation as indicated. By reciprocity consideration we identify (A+B) with the sum pattern, and (A - B) with the difference pattern. It is apparent that

$$(A+B) = (2a \cos\Psi) \quad \text{and}$$
$$(A-B) = i(2a \sin\Psi),$$

which suggests that the sum and difference voltages are derivable as components of $2a\epsilon^{i\Psi}$. It is interesting to note that, for the uniform case, the sum and difference patterns are derivable as the real and imaginary parts, respectively of

$$I \frac{\sin nu}{\sin u} \epsilon^{inu} = F_\sigma(u) + F_\Delta(u),$$

as previously defined.

As a means of comparing (A + B) and (A - B) let us consider their linear combination, viz

$$(A+B) + (A-B) = A'(u)\epsilon^{i\Psi(u)} = F_\sigma(u) + F_\Delta(u)$$

which we term the error signal.

Plots of the combined amplitude characteristics are shown in Fig.(5); plots of the combined phase characteristics are shown in Fig.(6) for the arrays previously used for typical calculation. Fig.(7) shows both the combined phase and amplitude characteristics of the eight-element array with optimum distribution for

252 deg. spacing.

It is apparent from these curves that large variations in resultant phase angle are obtainable at a sacrifice in amplitude, the intermediate values of 180 deg. occuring at points of zero amplitude. A significant feature of all the curves is that for $0 < |u| < \pi$ the phase angle does not change sign. Recalling the definition

$$u = \pi \frac{d}{\lambda} \sin\theta ,$$

we see readily that $u < \pi$ for $\frac{d}{\lambda} < 1$ for any θ, hence, we conclude that for spacing less than full wave, if an error signal of leading phase is associated with positive values of θ it follows that a lagging phase is associated with all negative values of θ. That is to say, no ambiguity exists in the error signal when $d/\lambda < 1$. In this regard it is pointed out that the phase characteristic has a negative slope at $u =$ 180 deg., hence this point, if it exists is a point of instability and does not constitute an ambiguity.

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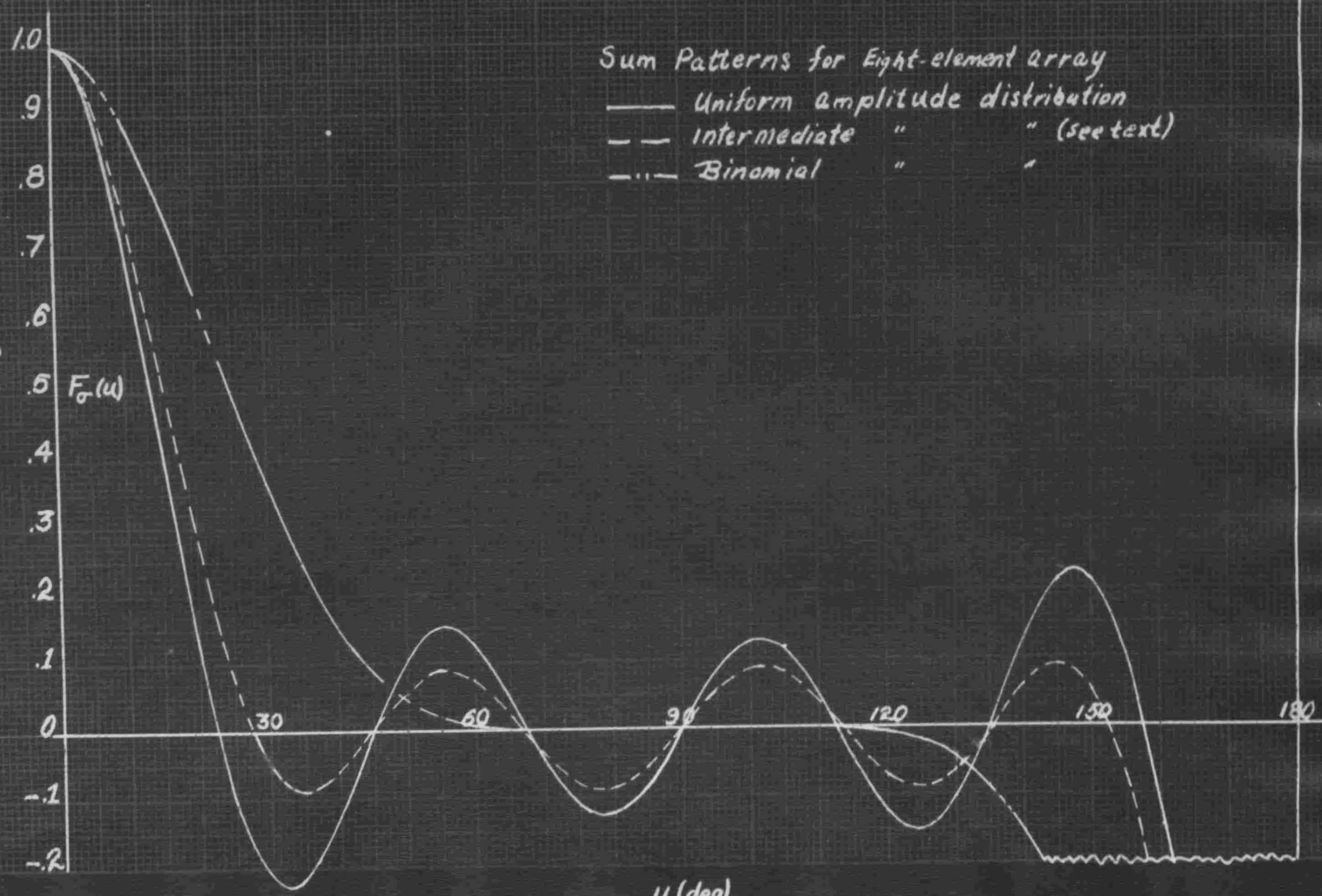
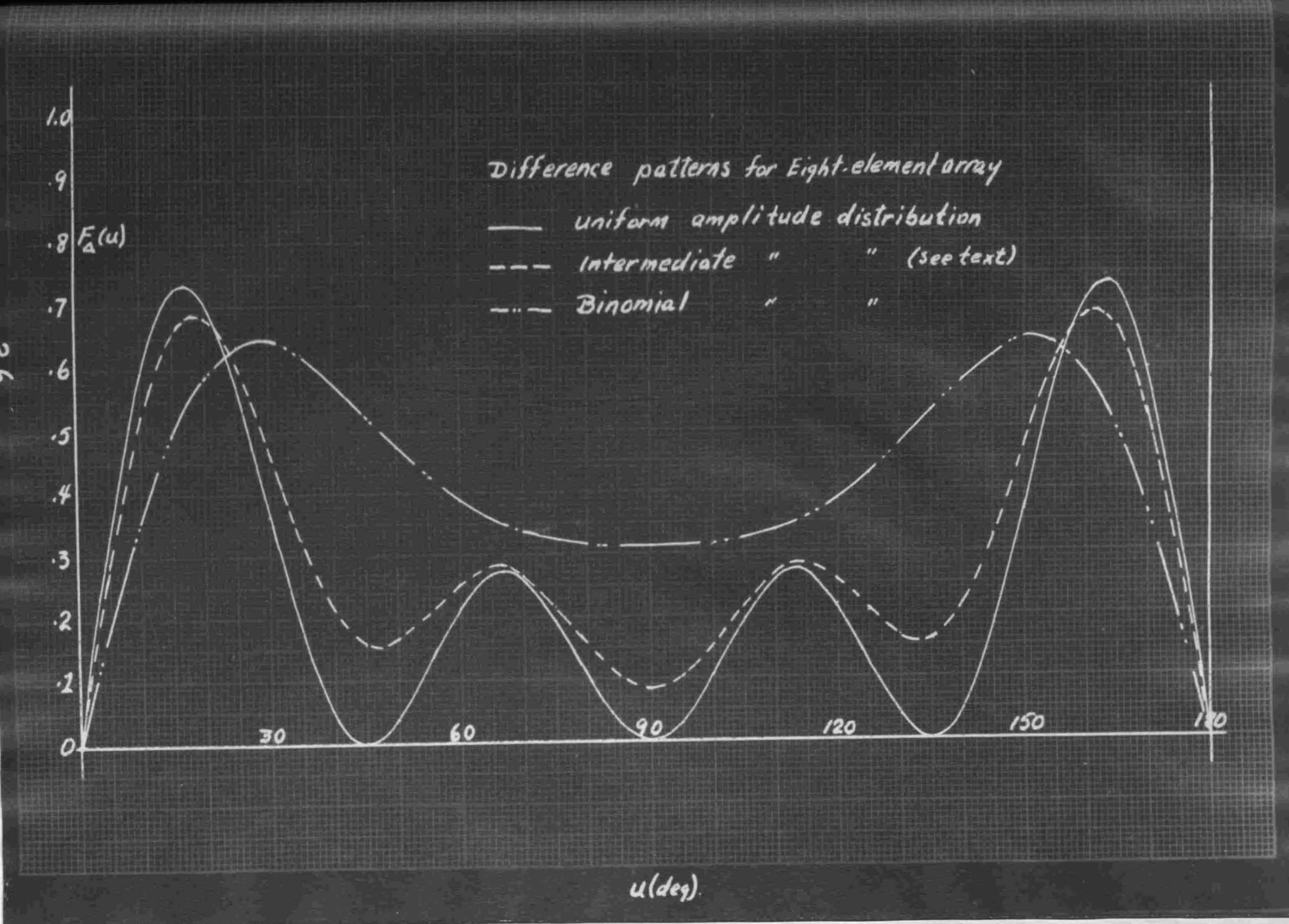
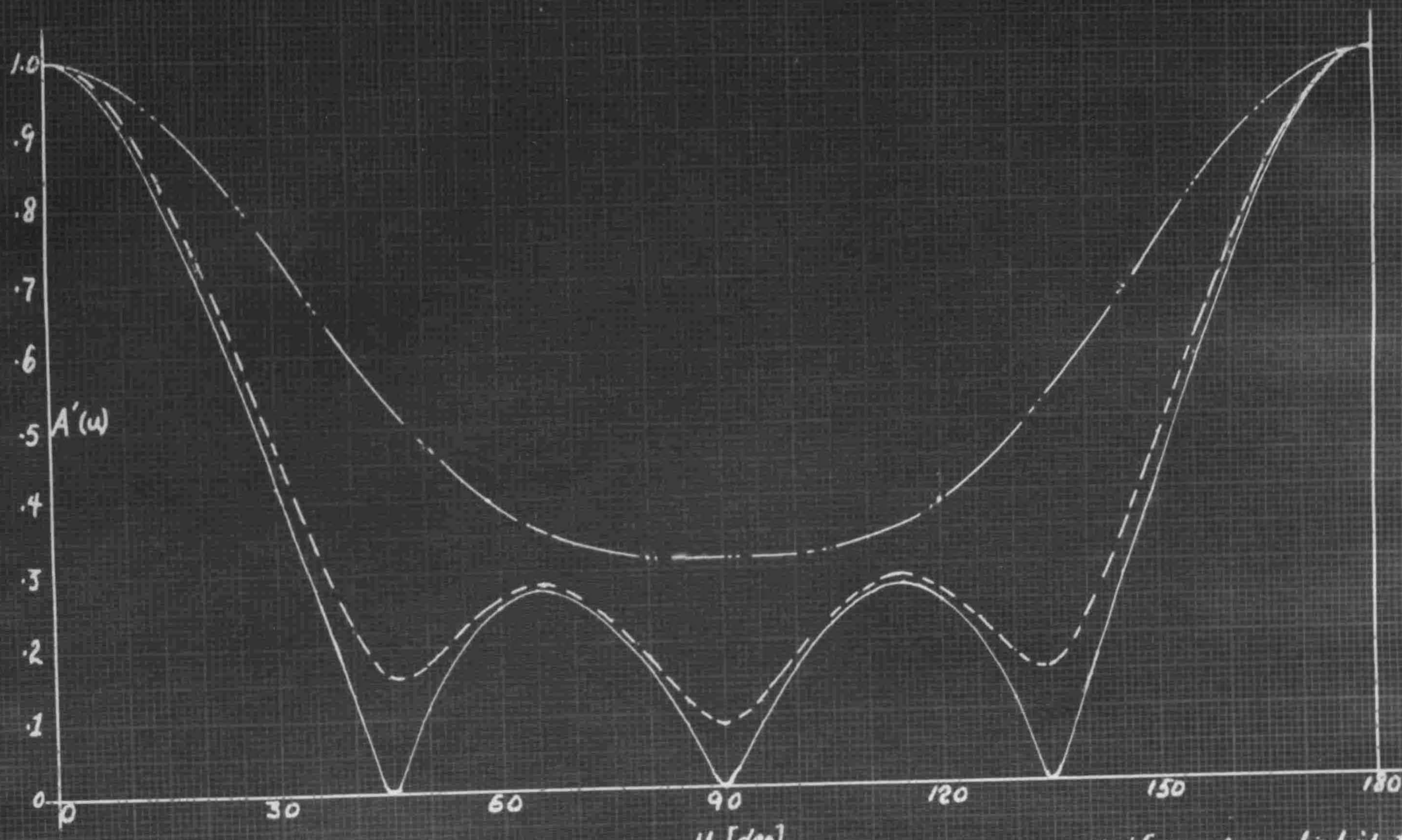

Sum Patterns for Eight-element array
Uniform amplitude distribution
Intermediate " " (see text)
Binomial " "
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
-.2
$F_\sigma(u)$
30
60
90
120
150
180
u (deg).

26

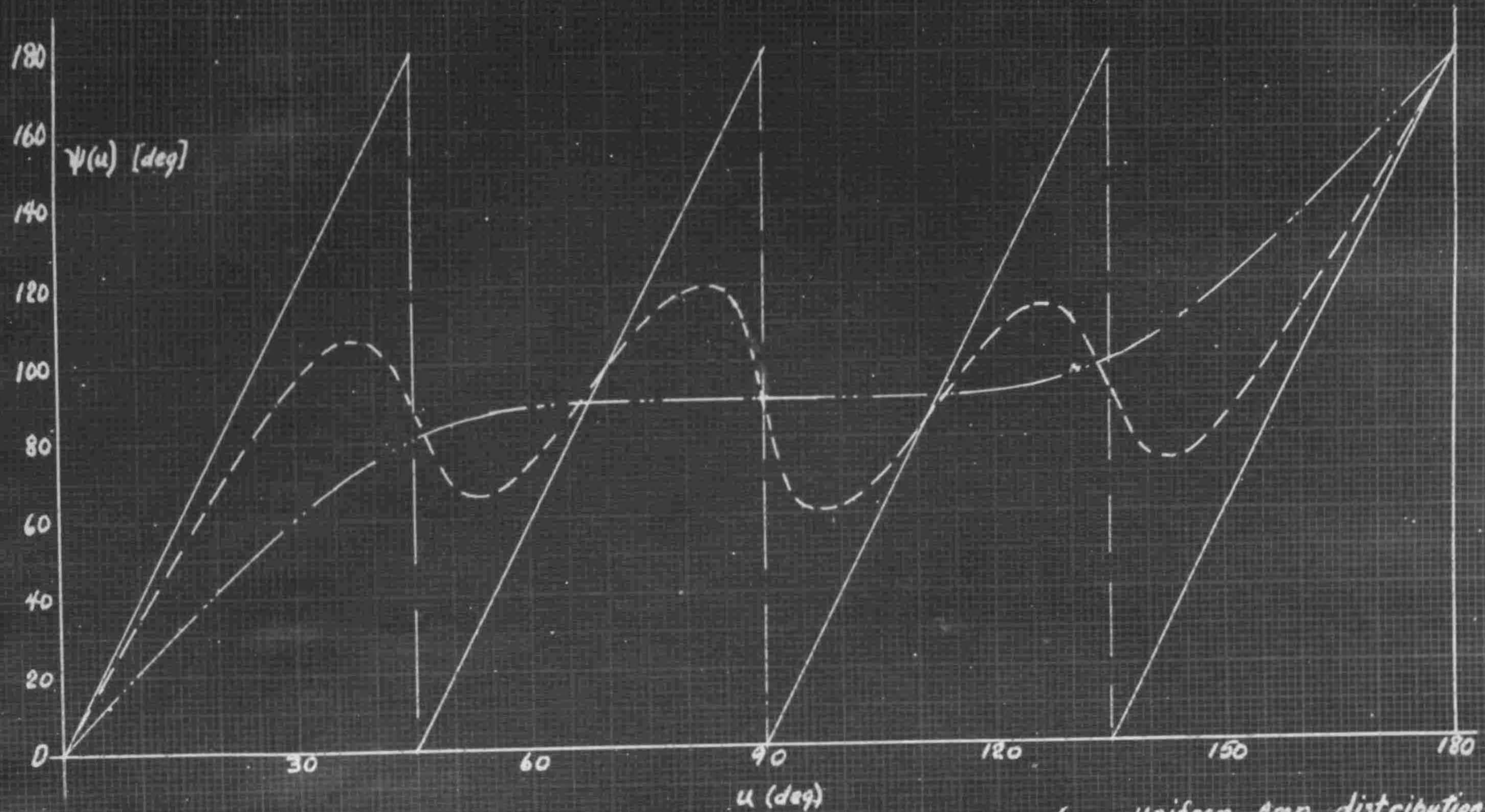


Difference patterns for Eight-element array
uniform amplitude distribution
Intermediate " " (see text)
Binomial " "
$F_\Delta(u)$
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
30
60
90
120
150
180
u(deg).



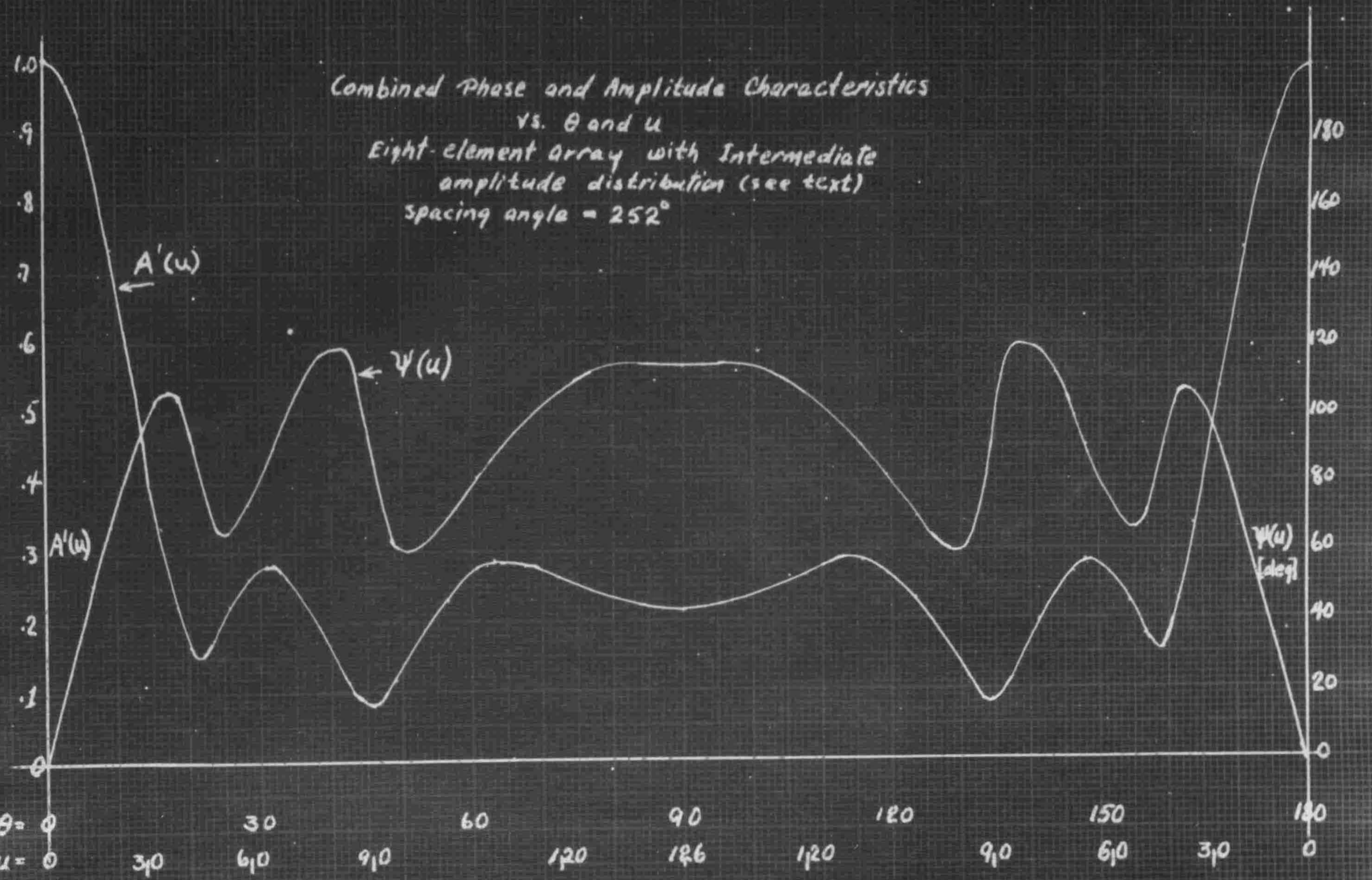
Combined Amplitude Characteristic for Eight-Element Array

— Uniform Amp. distribution
--- Intermediate " "
-·-·- Binomial " "



Combined phase characteristics for Eight element array

— uniform Amp. distribution
--- Intermediate " "
-·- Binomial " "



Combined Phase and Amplitude Characteristics
vs. θ and u
Eight-element array with Intermediate
amplitude distribution (see text)
Spacing angle = 252°
A'(u)
Ψ(u)
A'(u)
Ψ(u) [deg]
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
180
160
140
120
100
80
60
40
20
0
θ = 0
30
60
90
120
150
180
u = 0
30
60
90
120
126
120
90
60
30
0